Enhanced Convergence of Periodic Potentials in Stratified Media Through Asymptotic Extractions

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The efficient computation of periodic Green's functions is discussed here for an arbitrarily directed array of point sources in layered media. These Green's functions are necessary to formulate boundary integral equations for arrays of scatterers inside a general layered medium, solved with a method of moments approach in the spatial domain. For this reason, mixed-potential Green's functions –having a mild spatial singularity– are selected. The case of horizontally oriented dipoles (i.e., orthogonal to the stratification direction) is rather simple and has been solved previously; asymptotic terms are extracted that correspond to free-space Green's functions of periodic arrays of dipoles. On the other hand, the case of vertically-oriented dipoles (i.e., aligned perpendicular to the layers) is more intricate, since the extracted terms cannot be transformed into well-known Green's functions. A previous work dealt with arrays of line sources, while previous conference papers described the new approach for point sources, without providing analytical details. Applications to the dispersive analyses of leaky-wave antennas, not included here for the sake of brevity, will be presented at the conference.

Index Terms—**Periodic problems, layered media, mixed potentials, Green's functions, Ewald method, leaky-wave antennas**.

I. Introduction

THE EFFICIENT computation of Green's functions in complex environments can extend the applicability of the method of moments (MoM) as an attractive approach to minimize the number of unknowns in electromagnetic problems. The case of vertically stratified media is a good example of a canonical geometry where Green's functions can be easily defined; on the other hand, their calculation poses difficulties in terms of accuracy and computation time. Extensive work has been carried out for a single excitation, leading to different definitions of Green's functions [1]. Considerably less effort has been devoted to Green's functions due to periodic sources, required to study e.g. microstrip lines perturbed by arbitrarily shaped periodic loadings [2], [3] (see Fig. 1).

In the periodic case, with each source oriented orthogonal to the stratification direction, asymptotic extractions require the computation of homogeneous-medium periodic Green's functions for arrays of dipoles. This task is rather simple, since standard results are available in order to compute these functions, e.g. through the Ewald summation method [4]. On the other hand, if an array of vertical dipoles is considered (i.e., dipoles oriented along the stratification direction), the extracted terms are not Green's functions of arrays of dipoles, and standard computation techniques no longer hold.

In this paper, we focus on the computation of potentials due to vertical dipoles by means of suitable asymptotic extractions. A similar approach has been proposed by some of the authors for arrays of line sources [5], but the case of point sources, while presented at a few conferences [6], [7], has not been treated in full detail so far.

II. Extraction of Asymptotic Terms

Three potentials are required to study vertical metallic objects in layered media: $P_z^{\rm p}$ $G_{Azx}^{\rm p}$and $G_{Azy}^{\rm p}$ $G_{Azx}^{\rm p}$and $G_{Azy}^{\rm p}$ can be treated by differentiating the accelerated expressions obtained for $P_z^{\rm p}$, and thus will not be analyzed further [1]. $P_z^{\rm p}$ is expressed as a series of integrals if a 1-D periodic array of dipoles is considered (this case will be treated henceforth). Unfortunately, both the series and the integrals are very slowly (algebraically) converging when the observation and source point lie on the same interface between layers. To enhance their convergence, the following asymptotic extraction of quasi-static terms can be performed:

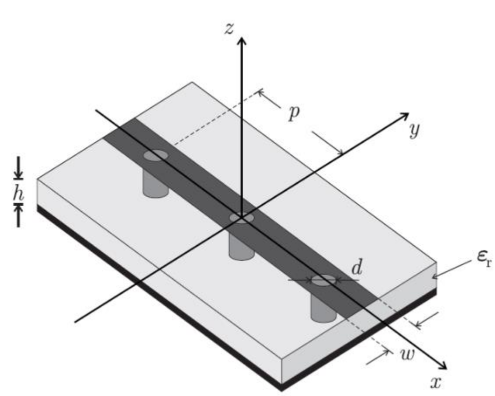

Fig. 1. A structure requiring the computation of vertical periodic potentials.

$$P_z^{\mathrm{p}}(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi p}\sum_{n=-\infty}^{+\infty} e^{-jk_{xn}\Delta x}\int_{-\infty}^{+\infty}\Big[\tilde{P}_z\left(k_y;z,z'\right) - \tilde{P}_z^{\infty}\left(k_y;z,z'\right)\Big]e^{-jk_y\Delta y}\mathrm{d}k_y + P_z^{\mathrm{p},\infty}(\mathbf{r},\mathbf{r}'), \quad (1)$$

where primed and unprimed coordinates refer to source and observation points, in the l' and l layers, respectively. p is the period, z is the stratification axis, and $k_{xn} = k_{x0} + 2\pi n/p$, k_{x0} being the Bloch-mode wavenumber. The expressions for the extracted term $\tilde{P}_z^{\infty}$ can be found in [6], and will not repeated here for the sake of brevity. We give for the first time the closed-form solution for the spatial domain $P_z^{\mathrm{p},\infty}$ term: it is a sum of potentials $g^{\mathrm{p},z}$ *due to an array of half-line sources*, computed by integrating the expressions for dipoles sources [5][4]. The integration can be performed analytically, resulting in

$$g^{\mathrm{p},z}(\mathbf{r},\mathbf{r}') = g^{\mathrm{p},z}_{\mathrm{spectral}}(\mathbf{r},\mathbf{r}') + g^{\mathrm{p},z}_{\mathrm{spatial}}(\mathbf{r},\mathbf{r}') \quad (2)$$

$$g^{\mathrm{p},z}_{\mathrm{spectral}} = \frac{1}{4\sqrt{\pi}Ep}\sum_{n=-\infty}^{+\infty} e^{-jk_{xn}\Delta x}\int_{1}^{\infty}\mathrm{erfc}\left(\frac{|\Delta z|E}{u}\right)e^{-\frac{\Delta y^2E^2}{u^2}+\frac{k_{\rho n}^2u^2}{4E^2}}\mathrm{d}u \quad (3)$$

with $k_{\rho n} = \sqrt{k^2 - k_{xn}^2}$, and

$$g^{\mathrm{p},z}_{\mathrm{spatial}} = \frac{1}{4\pi}\sum_{n=-\infty}^{+\infty} e^{-jnk_{x0}p}\int_{E}^{\infty}\mathrm{erfc}\left(|\Delta z|s\right)\frac{e^{-R_{tn}^2s^2+\frac{k^2}{4s^2}}}{s}\mathrm{d}s \quad (4)$$

with $R_{tn} = \sqrt{(\Delta x - np)^2 + \Delta y^2}$, and E is a suitable Ewald splitting parameter defined in [4, p. 254, (28), (31)-(34)].

The two series (3) and (4) present a very fast (Gaussian) convergence as in the original Ewald series. Furthermore, the series converge even in the case of improper waves, i.e. waves not respecting the boundary conditions at infinity, often necessary to describe leaky waves [2].

III. Numerical Results

In Fig. 2, we show the comparison between the unaccelerated and the accelerated Green's function P_z^{p} along the segment $0 < \Delta y < p$, $\Delta x = p/2$. The agreement between the curves verifies the correctness of the acceleration procedure.

In Fig. 3, we show the decay of the terms in the series for two values of the spatial period p, with and without the extraction of $\tilde{P}_z^{\infty}$. The enhanced convergence rate is clearly visible by analyzing the slope of the curves (from n^{-2} to n^{-4}).

IV. Conclusions

We have proposed an asymptotic extraction of quasi-static images for mixed-potential Green's functions related to vertical current elements in layered media. The extracted terms are identified as potentials due to arrays of half-line currents, and are computed accordingly. Numerical results demonstrated the accuracy of the acceleration scheme and the enhanced convergence of the spectral series.

V. Acknowledgment

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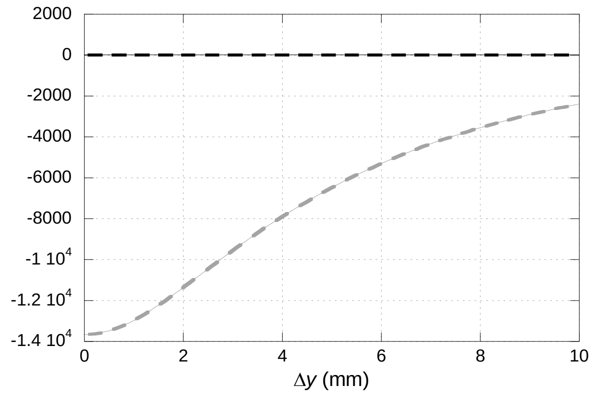

Fig. 2. P_z^{p} with (*continuous lines*) and without (*dashed lines*) acceleration: real parts (*black lines*) and imaginary parts (*gray lines*). The structure is a grounded slab in air, with $\varepsilon_{\mathrm{r}} = 10.2$, $\mu_{\mathrm{r}} = 3$, $h = 0.767$ mm, $p = 10$ mm, at a frequency $f = 10$ GHz. P_z^{p} is computed along the segment $0 < \Delta y < p$, $\Delta x = p/2$, and $z = z' = 0$ (the interface between the slab and the air). The Bloch wavenumber is $k_{x0} = 1.5k_0$. (Note: the term P_z^{p} is unitless.)

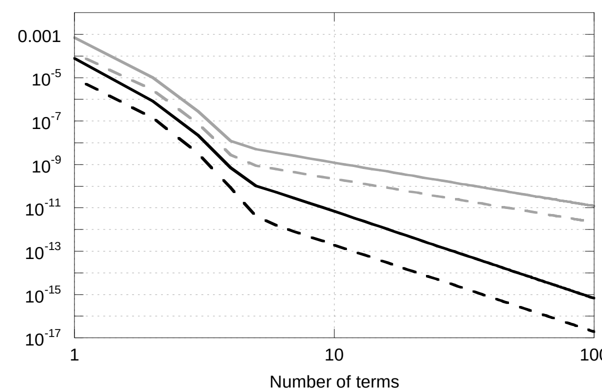

Fig. 3. Magnitude of the aggregated (n,$-n$) terms of the unaccelerated (*gray lines*) and the accelerated (*black lines*) Green's functions, vs. the summation index n. The structure is that of Fig. 2, but two periods are chosen: $p = 10$ mm (*solid lines*), $p = 4$ mm (*dashed lines*). The Bloch wavenumber is $k_{x0} = (0.8-j0.1)k_0$. Coordinates: $z = z' = 0$ (air-substrate interface), $\Delta x = \Delta y = p/2$.

VI. References

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